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GLOBALLY UNIVALENT C1-MAPS WITH SEPARABILITY.(U)

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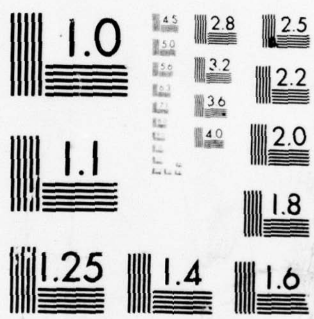

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GLOBALLY UNIVALENT C^1-MAPS WITH SEPARABILITY

Masakazu Kojima and Michael J. Todd

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ABSTRACT

Let f be a continuously differentiable (C^1) map from a compact rectangular region Q in the n-dimensional Euclidean space R^n into R^n. Gale and Nikaido showed that if all the principal minors of the Jacobian of $f:Q \to R^n$ are positive everywhere in Q then f is univalent (one-to-one) on Q. This result was recently strengthened and extended to a general case where f is defined on a compact convex polyhedron $P \subset R^n$ with nonempty interior by Mas-Colell and others. In this paper, we deal with a special case where f is separable with respect to a subset of variables, i.e.,

$$f(x) = \sum_{i=1}^{m} f^i(x_i) + f^{m+1}(x_{m+1},\ldots,x_n)$$

$$\text{for all } x = (x_1,\ldots,x_n) \in P = P_1 \times P_2 .$$

Here P_1 is a compact rectangular region in R^m, P_2 a compact polyhedron in R^{n-m}, f^i a C^1-map from an interval of the real line into R^n and $f^{m+1}:P_2 \to R^n$ a C^1-map. We show that the sufficient condition given by Mas-Colell for a C^1-map $f:P \to R^n$ to be univalent can be weakened in this case.

AMS(MOS) Subject Classification - 65H10, 57D50, 57D35

Key Words: Jacobian Matrix, Global Univalence, Homeomorphism, Separable Maps, Systems of Equations

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SIGNIFICANCE AND EXPLANATION

When solving one equation in one unknown, $f(x) = q$, it is obvious geometrically that if $f(x)$ is continuously differentiable and $f'(x) \neq 0$ for all x, then for each q the equation has at most one solution (f is then said to be univalent). Of course the univalence of f does not ensure the existence of a solution, for example, $e^x = 0$. e to the x power

When solving a system of n equations in n unknowns,

del f sub i / del x sub k

$$(*) \qquad f_i(x_1,\ldots,x_n) = q_i \qquad (i = 1,\ldots,n),$$

the analogue of $f'(x)$ is the $n \times n$ Jacobian matrix $[\partial f_i/\partial x_k]$. It is interesting to investigate conditions on the Jacobian matrix which will ensure the univalence of the left hand of the equation ~~(*)~~. Such conditions are of practical importance if they are combined with conditions which ensure the existence of solutions because if the equation ~~(*)~~ has a solution and if the left hand of the equation ~~(*)~~ is univalent then the solution is unique.

In this paper we deal with the case where $f_i(x_1,\ldots,x_n)$ can be written in the form

$$f_i(x_1,\ldots,x_n) = \sum_{j=1}^{m} f_i^j(x_j) + f_i^{m+1}(x_{m+1},\ldots,x_n) \qquad (i=1,\ldots,n),$$

and give a condition on the Jacobian matrix which ensures the univalence. As a special case, it is shown that if $m = n-1$ and if the determinant of the Jacobian matrix $[\partial f_i/\partial x_k]$ is nonzero for all $(x_1,\ldots,x_n)$ then the left hand of (*) is univalent.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

GLOBALLY UNIVALENT C^1-MAPS WITH SEPARABILITY

Masakazu Kojima and Michael J. Todd

1. Introduction

Let R^n denote the n-dimensional Euclidean space. We say that a subset Q of R^n is a rectangular region if

$$Q = \{x = (x_1,\dots,x_n) \in R^n : a_i \leqq x_i \leqq b_i \ (i = 1,\dots,n)\},$$

where $-\infty \leqq a_i < b_i \leqq +\infty$ $(i = 1,\dots,n)$. We call a rectangular region in the real line R an interval. More than a decade ago Gale and Nikaido [1] showed that if all the principal minors of the Jacobian of a continuously differentiable (C^1) map f from a rectangular region Q in R^n into R^n are positive everywhere in Q then f is univalent (one-to-one) on Q. Recently, this result was strengthened and generalized by Garcia and Zangwill [2] and Mas-Colell [4]. This paper has a close relation with Mas-Colell's generalization [4].

We say that a convex set $C \subset R^n$ spans a subspace L of R^n if

$$L = \{\lambda(x-x^0) : \lambda \in R,\ x \in C\}$$

for some relative interior point x^0 of C. It should be noted that the set $\{\lambda\ (x-x^0) : \lambda \in R,\ x \in C\}$ does not depend on the choice of an interior point x^0 of C. Let L be a nonempty subspace of R^n. We denote the orthogonal projection map from R^n onto L by $\Pi_L : R^n \to L$; $\Pi_L(x) \in L$ and $\|x - \Pi_L(x)\| = \min\{\|x-y\| : y \in L\}$ for every $x \in R^n$. Let M be an $n \times n$ matrix. Then the composite map $\Pi_L \circ M : L \to L$ is linear. Suppose $\dim L = k$ and that the set of the columns of an $n \times k$ matrix A forms a basis of L. Then we can write

$$\Pi_L(x) = A(A^TA)^{-1}A^Tx \quad \text{for every } x \in R^n,$$

where A^T denotes the transpose of the matrix A. It is easily verified that the linear map $\Pi_L \circ M : L \to L$ has a positive (or negative) determinant if and only if the $k \times k$ matrix A^TMA has a positive (or negative) determinant. The positivity (or negativity) of the determinant of the linear map $\Pi_L \circ M : L \to L$ does not depend on the choice of a basis of L. We denote the Jacobian matrix of a C^1-map f from a subset of R^m into R^n by $Df(x)$.

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Theorem 1 (Theorem 1 in Mas-Colell [4]): Let P be a compact convex polyhedron in R^n with nonempty interior, and $f:P \to R^n$ be a C^1-map. Assume that for every $x \in P$ and subspace $L \subset R^n$ spanned by a face σ of P which includes x, the linear map $\Pi_L \circ Df(x):L \to L$ has a positive determinant. Then f is univalent on P.

We consider a special case where a C^1-map $f:P \to R^n$ is separable with respect to some of the variables and show that the assumption of Theorem 1 can be weakened in this case.

Theorem 2: Let $P_1 \subset R^m$ be a compact rectangular region and $P_2 \subset R^{n-m}$ a compact convex polyhedron with nonempty interior. Let $f : P_1 \times P_2 \to R^n$ be a C^1-map such that

$$f(x) = \sum_{i=1}^{m} f^i(x_i) + f^*(x_{m+1},\dots,x_n)$$

for every $x = (x_1,\dots,x_n) \in P_1 \times P_2$, where f^i is a C^1-map from an interval of R into R^n and f^* is a C^1-map from P_2 into R^n. Assume that for every face τ_2 of P_2 and every $x \in P_1 \times \tau_2$, the linear map $\Pi_L \circ Df(x):L \to L$ has a positive determinant, where L is the subspace spanned by $P_1 \times \tau_2$. Then f is univalent on $P_1 \times P_2$.

We will derive Theorem 2 from Theorem 1 in Section 2. When $m = 0$, Theorems 1 and 2 are equivalent. Suppose $m \geqq 1$. Note that each face of $P_1 \times P_2$ is of the form $\tau_1 \times \tau_2$, with τ_i a face of P_i $(i=1,2)$; we do not require that $\Pi_L \circ Df(x):L \to L$ have a positive determinant if the subspace L is spanned by a face $\tau_1 \times \tau_2$, $\tau_1 \neq P_1$. Hence the hypothesis of Theorem 2 is weaker than that of Theorem 1.

It is well-known that the positivity of a C^1-map $f:P \to R^n$ does not necessarily ensure the univalence of f. For example, tha map $f:R^2 \to R^2$ defined by

$$f_1(x_1,x_2) = (\exp x_1)(\sin x_2)$$
$$f_2(x_1,x_2) = -(\exp x_1)(\cos x_2)$$

has a positive Jacobian at every $x = (x_1,x_2) \in R^2$, but it is not univalent. When the dimension n is equal to 1 or $f:P \to R^n$ is affine, however, the positivity of the Jacobian at every $x \in P$ implies the univalence. These two exceptional cases are unified by the following result.

Corollary: Let $Q \subset R^n$ be a rectangular region and $f:Q \to R^n$ a C^1-map such that

$$f(x) = \sum_{i=1}^{m} f^i(x_i) \quad \text{for all } x \in Q,$$

where f^i is a C^1-map from an interval of R into R^n. Assume that the Jacobian of the

map f is nonzero at every $x \in Q$. Then f is univalent on P .

<u>Proof</u>. Let $x^0 \in Q$. Then $f:Q \to R^n$ is univalent if and only if the map $g:Q \to R^n$ defined by

$$g(x) = Df(x^0)\ f(x) \quad \text{for all} \quad x \in Q$$

is univalent. Obviously $g:Q \to R^n$ is separable with respect to all the variables and $\det Dg(x) > 0$ for all $x \in Q$. By Theorem 2, we see that g is univalent on every compact rectangular region contained in Q . This implies g is univalent on Q .

Q. E. D.

2. Proof of Theorem 2.

Throughout this section e^i denotes the i-th unit vector in R^n, and I the identity matrix of appropriate dimension. For simplicity of notation, we assume that $0 \in R^n$ is a vertex of $P_1 \times P_2$. Let $M = Df(0)$. We partition the $n \times n$ matrix M such that

$$M = \begin{array}{c} \begin{array}{cc} m & n-m \end{array} \\ \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array} \right] \end{array} \begin{array}{l} \\ m \\ n-m \end{array} .$$

Let E denote the $n \times m$ matrix $[e^1, \ldots, e^m]$. Since $0 \in R^n$ lies in a face $P_1 \times \{0\}$ of $P_1 \times P_2$ and the set of the columns of E forms a basis of the subspace $R^m \times \{0\} \subset R^n$ spanned by $P_1 \times \{0\}$, we have

$$\det M_{11} = \det E^T Df(0)E > 0 .$$

Define the $n \times n$ matrix

$$N = \begin{bmatrix} M_{11}^{-1} & 0 \\ -M_{21}M_{11}^{-1} & I \end{bmatrix} ,$$

and the map $g: P_1 \times P_2 \to R^n$ by

$$g(x) = N f(x) \quad \text{for all} \quad x \in P_1 \times P_2 .$$

Obviously $g: P_1 \times P_2 \to R^n$ is also separable with respect to the variables $x_1, \ldots, x_m$. That is, we can write

$$g(x) = \sum_{i=1}^{m} g^i(x_i) + g^*(x_{m+1}, \ldots, x_n) \quad \text{for all} \quad x \in P_1 \times P_2 ,$$

where g^i $(i = 1, \ldots, m)$ and g^* are C^1-maps. Since the $n \times n$ matrix N is nonsingular, f is univalent on $P_1 \times P_2$ if and only if g is univalent on $P_1 \times P_2$. We shall establish that g is univalent on $P_1 \times P_2$.

In view of Theorem 1, it suffices to show that for every $x \in P_1 \times P_2$ and subspace $L \subset R^n$ spanned by a $(k+\ell)$-dimensional face $\tau_1 \times \tau_2$ of $P_1 \times P_2$ which includes x and for an $n \times (k+\ell)$ matrix A whose columns form a basis of L, the determinant of $A^T Dg(x)A$ is positive. Assume that τ_1 and τ_2 have dimensions k and ℓ resepctively. Since τ_1 is a face of the compact rectangular region $P_1 \subset R^m$, we can choose k vectors

from the m unit vectors $e^1,\dots,e^m$ in R^n for a basis of the subspace L_1 spanned by $\tau_1 \times \{0\} \subset R^n$. For simplicity of notation, we assume that the last k unit vectors $e^{m-k+1},\dots,e^m$ form a basis of the subspace L_1. Choose a set of ℓ vectors $u^1,\dots,u^\ell$ for a basis of the linear subspace spanned by $\{0\} \times \tau_2 \subset R^n$. Define the $n \times (k+\ell)$ matrix

$$A = [e^{m-k+1},\dots,e^m,u^1,\dots,u^\ell].$$

By the construction, the set of the columns of the $n \times (k+\ell)$ matrix forms a basis of the subspace L. The purpose of the remainder of the proof is to show

$$\det A^T Dg(x)A > 0 .$$

Let $y = (0,\dots,0,x_{m-k+1},\dots,x_n) \in R^n$. Then y lies in a face $P_1 \times \tau_2$ of $P_1 \times P_2$, and the columns of the $n \times (m+\ell)$ matrix

$$\bar{A} = [e^1,\dots,e^m,u^1,\dots,u^\ell]$$

form a basis of the subspace $\bar{L}$ spanned by $P_1 \times \tau_2$. By assumption, we have

$$\det \bar{A}^T Df(y)\bar{A} > 0 .$$

By the construction of $g\colon P_1 \times P_2 \to R^n$, we see

$$Dg(0) = NDf(0) = \begin{bmatrix} I & M_{11}^{-1}M_{12} \\ & \\ 0 & -M_{21}M_{11}^{-1}M_{12}+M_{22} \end{bmatrix}.$$

Hence it follows from the separability of the map $g\colon P_1 \times P_2 \to R^n$ that

$$Dg^i(0) = e^i \qquad (i = 1,2,\dots,m).$$

Let B denote the $n \times (m-k)$ matrix $[e^1,\dots,e^{m-k}]$. Then

$$Dg(y) = [B,Dg^{m-k+1}(x_{m-k+1}),\dots,Dg^m(x_m),Dg^*(x_{m+1},\dots,x_n)],$$

$$\bar{A} = [B,A]$$

and

$$\bar{A}^T Dg(y)\bar{A} = \begin{bmatrix} B^T Dg(y)B & B^T Dg(y)A \\ & \\ A^T Dg(y)B & A^T Dg(y)A \end{bmatrix} .$$

It is easily verified that $B^T Dg(y)B = I$, $A^T Dg(y)B = 0$ and $A^T Dg(y)A = A^T Dg(x)A$. Hence

$$\det A^T Dg(x)A = \det \bar{A}^T Dg(y)\bar{A} .$$

If we write the $n \times (m+\ell)$ matrix $\bar{A}$ as

$$\bar{A} = \begin{bmatrix} I & 0 \\ & \\ 0 & A_{22} \end{bmatrix} ,$$

where A_{22} is an $(n-m) \times \ell$ matrix, then we see

$$\bar{A}^T Dg(y)\bar{A} = \bar{A}^T NDf(y)A$$

$$= \begin{bmatrix} M_{11}^{-1} & 0 \\ & \\ -A_{22}^T M_{21} M_{11}^{-1} & I \end{bmatrix} \bar{A}^T Df(y)\bar{A} .$$

Recalling that $\det M_{11} > 0$, we consequently obtain $\det A^T Dg(x)A = \det \bar{A}^T Dg(y)\bar{A} =$ $(\det M_{11}^{-1}) \det \bar{A}^T Df(y)\bar{A} > 0$. Q.E.D.

3. Concluding Remark.

Recently, Kojima and Saigal extended Theorem 1 to the case where the map $f: P \to R^n$ is piecewise continuously differentiable ([Theorem 4.3, 3]). Theorem 2 can be also extended to a piecewise continuously differentiable case. The same proof as we have given in Section 2 is valid for the extension if we use Theorem 4.3 of [3] instead of Theorem 1.

REFERENCES

[1] D. Gale and H. Nikaido, "The Jacobian Matrix and the Global Univalence of Mappings," Math. Ann. 159 (1965), 81-93.

[2] C. B. Garcia and W. I. Zangwill, "On Univalence and P-Matrices," Report 7737, Center for Mathematical Studies in Business and Economics, University of Chicago, July 1977.

[3] M. Kojima and R. Saigal, "A Study of PC^1 Homeomorphisms on Subdivided Polyhedrons," Technical Report, Northwestern University, April 1978.

[4] A. Mas-Colell, "Homeomorphisms of Compact Convex Sets and the Jacobian Matrix," Technical Report, Universität Bonn, April 1977, submitted to SIAM J. Applied Math.

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Let f be a continuously differentiable (C^1) map from a compact rectangular region Q in the n-dimensional Euclidean space R^n into R^n. Gale and Nikaido showed that if all the principal minors of the Jacobian of $f:Q\to R^n$ are positive everywhere in Q then f is univalent (one-to-one) on Q. This result was recently strengthened and extended to a general case where f is defined on a compact convex polyhedron $P \subset R^n$ with nonempty interior by Mas-Colell and others. In this paper, we deal with a special case where f is separable with respect to a subset of variables, i.e.,

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Abstract (continued)

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Here P_1 is a compact rectangular region in R^m, P_2 a compact polyhedron in R^{n-m}, f^i a C^1-map from an interval of the real line into R^n and $f^{m+1}:P_2 \to R^n$ a C^1-map. We show that the sufficient condition given by Mas-Colell for a C^1-map $f:P \to R^n$ to be univalent can be weakened in this case.